interface deformation is shown): *1* is the total wave arising under the action on the film surface of normal and tangential stresses, *2* is the wave arising under the action of normal stresses only, $\eta_{20} = 0.01\xi$. The parameter values are $\delta = 0.93$; $\lambda = 4.99$, $a = 5$, $\beta_1 = 12.3$, $\beta_2 = 0.0025$, $\gamma = 0.0037$, $R = 500$, $F = 1.09$. In this case the thin film reduces the wave amplitude by one another.

REFERENCES

1. LAMB H., Hydrodynamics, Dover, 1932.
2. SRETENSKII L.N., Theory of Wave Motions of Fluid (Teoriya volnovykh divizhenii zhidkosti), Nauka, Moscow, 1977.
3. CHERKESOV L.V., Hydrodynamics of Surface and Interior Waves (Gidrodinamika poverkhnostnykh i vnutrennykh voln), Nauk. Dumka, Kiev, 1976.
4. KOCHIN N.E., KIBEL' I.A. and ROZE N.V., Theoretical hydormechanics (Teoreticheskaya gidromekhanika), Fizmatgiz, Moscow, 1963.
5. KOCHIN N.E., KIBEL' I.A. and ROZE N.V., Theoretical Hydromechanics (Teoreticheskaya gidromekhanika), Gostekhizdat, Moscow-Leningrad, 1948.
6. GRADSHTEIN I.S. and RYZHIK I.M., Tables of Integrals, Sums, Series, and Products (Tablitsy integralov, summ, ryadov i proizvedenii), Nauka, Moscow, 1971.

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A HYDRODYNAMIC MODEL OF AN INTERCEPTING VACUUM DRAINAGE IN A DESCENDING FLOW OF GROUND WATER*

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A boundary value problem is formulated describing two-dimensional steady filtration in a layer of soil of infinite capacity, towards a horizontal vacuum drainage which captures partly or wholly, at some specified rate of drainage, the ground water filtering downwards from the soil surface. A solution of the problem is constructed using conformal mapping and the solution contains two unknown mapping parameters. A system of equations is derived for the latter parameters, and their unique solvability is established analytically. At the same time, a restriction on the filtration capacity of the drainage is revealed, corresponding to the critical mode (complete interception of the flow). A computer program is written for the algorithm used to compute, for the given parameters of the medium, the flow characteristics in the critical mode, and at some value of the drainage output chosen arbitrarily from the interval of admissible values. Numerical examples are given.

Horizontal vacuum drains are installed in the irrigated soil of infinite capacity. Their purpose is to intercept the irrigation water seeping through the root system from the surface, flooded with a thin layer of water. We will assume that every stream associated with the action of one or another drain is symmetrical about a vertical line passing through the centre of the drain. The side boundaries of the streams are free, and atmospheric pressure is maintained along them. For this reason the flows towards separate drains are themselves separate, so that we can speak of a system of drains in the model in question only in a conventional manner.

The right half of the zone of action of one of the drains represented by a point drain B (sink), is shown schematically in the $z = x + iy$ -variable plane as a region of flow in Fig.1a for the case when a proportion Q of the total (within the shaded region) inflow Q_s of surface water is intercepted by the drain, and the remainder Q_0 flows down to infinity. The corresponding domain of the complex potential $\omega = \varphi + i\psi$ is shown in Fig.1b. The investigation is carried

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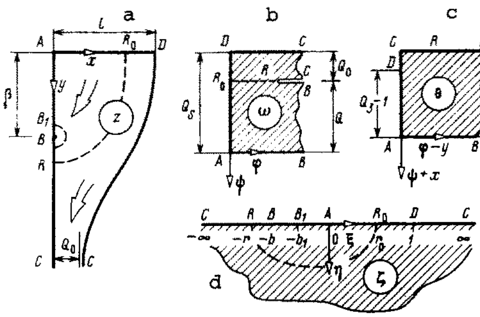

Fig.1

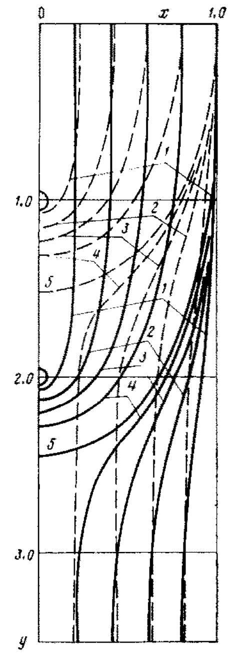

Fig.2

out in terms of the reduced quantities z and ω connected with the corresponding real quantities z_r and ω_r by the relations ($\varkappa$ is the coefficient of filtration)

$$z = z_r/l, \quad \omega = \omega_r/(\varkappa l) \tag{1}$$

Let us bring into our discussion the Zhukovskii function $\theta = \omega + iz$ whose domain is shown in Fig.1c. Mapping the regions ω and θ conformally onto the half-plane $\operatorname{Im}\zeta \geqslant 0$ (Fig.1d), we obtain

$$\omega = -i\frac{Q}{\pi}\frac{\sqrt{b(1+b)}}{r-b}\int_0^\zeta \frac{(r+u)\,du}{(b+u)\sqrt{u(1-u)}} = \tag{2}$$

$$-i\frac{2Q}{\pi}\left[\operatorname{arctg}\sqrt{\frac{(1+b)\zeta}{b(1-\zeta)}} + \frac{\sqrt{b(1+b)}}{r-b}\arcsin\sqrt{\zeta}\right]$$

$$\theta = -i\frac{Q\sqrt{b}}{\pi}\int_0^\zeta \frac{du}{(b+u)\sqrt{u}} = -\frac{2Q}{\pi}\operatorname{arctg}\sqrt{\frac{\zeta}{b}}, \quad z = i(\omega - \theta)$$

$$\frac{1}{w} = \frac{dz}{d\omega} = -i\frac{d\theta}{d\omega} + i = -i\frac{r-b}{\sqrt{1+b}}\frac{\sqrt{1-\zeta}}{\zeta+r} + i \tag{3}$$

where $w = w_x - iw_y$ is the complex rate of filtration /1/.

At the point D where $\zeta = 1$, $\omega = -iQ_s$, $z = 1$ (Fig.1a,b,d), relations (2) yield

$$Q_s = Q + Q_0, \quad Q_0 = \frac{\sqrt{b(1+b)}}{r-b}Q = 1 - \frac{2Q}{\pi}\operatorname{arctg}\sqrt{b} \tag{4}$$

Assuming that $z = i\beta$, $\zeta = -b$ we obtain from (2)

$$\beta = (Q/\pi)\ln(1+b) + (2Q_0/\pi)\operatorname{arsh}\sqrt{b} \tag{5}$$

Aiming at a straightforward formulation, we shall assume that the geometrical parameters of the scheme l, β and the drain flow rate Q are given for each specific case. As in /2/, we will study the flow pattern by varying Q for fixed l and β. Here the key problem is that of the form of the relation $b(Q)$ determined by Eq.(5). From it we obtain

$$\frac{db}{dQ} = \frac{4}{\pi}f(\sqrt{b})\left[\frac{Q}{1+b}\left(1 - \frac{2}{\pi}\frac{\operatorname{arsh}\sqrt{b}}{\sqrt{b}}\right) + \frac{Q_0}{\sqrt{b(1+b)}}\right]^{-1} \tag{6}$$

$$f(x) = \operatorname{arsh} x \cdot \operatorname{arctg} x - (\pi/4)\ln(1+x^2)$$

When a proportion of water filtering from the surface is intercepted by the drain, we have $Q > 0$, $Q_0 > 0$. Since we also have $\operatorname{arsh}\sqrt{b} < \sqrt{b}$ when $b > 0$, the expression within the square brackets on the left-hand side of (6) is also positive.

To find the sign of $f(\sqrt{b})$, we shall consider the following system of auxiliary functions:

$$f(x),\ f_1(x) = \frac{1+x^2}{x} f'(x),\quad f_2(x) = x^2\sqrt{1+x^2} f_1'(x),\quad f_3(x) = \frac{\sqrt{1+x^2}}{x} f_2'(x) \tag{7}$$

From the definition of the function $f(x)$ and relations (7) it follows that $f(0) = f_2(0) = f_3(0) = 0, f_{1,2,3}' < 0$ when $x > 0$. Furthermore, since $f_1(0) = 2 - \pi/2 > 0, f_1(\infty) = 0$ we find that $f_1(x) > 0, f'(x) > 0$ and finally $f(x) > 0$ when $x > 0$. As a result we conclude that

$$db/dQ > 0 \tag{8}$$

Thus the parameter b given by relation (5) for fixed β as a function of the flow rate Q, increases monotonically as Q increases. When Q reaches a certain value Q^*, we have

$$1 - (2Q^*/\pi)\operatorname{arctg}\sqrt{b^*} = 0,\quad b^* = b\,|_{Q=Q^*} \tag{9}$$

When relation (9) holds, we find from (4) that $r = \infty, Q_0 = 0, Q = Q_s$: the flow is fully intercepted by the drain and the points R and C coincide (Fig.1a, d). Relation (3) now becomes

$$1/w = i(1-\mu),\quad \mu = [(1-\zeta)/(1+b^*)]^{1/2} \tag{10}$$

Taking (10) and relations $w = d\omega/dz$ into account we find in the limiting case in question, under the drain sink B where $z = iy\ (y > \beta)$, $w = -iw_y$, $\omega = \varphi + iQ = -p/\gamma + y + iQ$ (p is the pressure and γ is the specific gravity of water), that

$$w_y = -\frac{1}{\gamma}\frac{dp}{dy} + 1 = \frac{1}{1-\mu},\quad \frac{dp}{dy} = \frac{\gamma}{1-\mu^{-1}} \tag{11}$$

On a separate segment $\zeta < -b$ and in accordance with (11), we obtain $w_y \geqslant 0$, $dp/dy \geqslant \gamma$ and the equalities hold only at the point R ($\zeta = -\infty$). At this point the pressure gradient counterbalances the force of gravity and overcomes it on the last part of the segment BR where the flow runs upwards. Increasing the vacuum by an additional arbitrarily small amount will upset the dynamic equilibrium between the flow and the atmosphere, and will cause a leakage of air into the drain from below. It therefore follows that the flow mode under discussion is critical. Complex filtration models with suspended flow were studied in /2, 3/ and the scheme of flow studied here was mentioned in the latter paper.

Eliminating Q from (5) using (9), we obtain the equation for determining the value b^* of the parameter b in the critical mode

$$F(b^*) = \frac{\ln(1+b^*)}{2\operatorname{arctg}\sqrt{b^*}} = \beta \tag{12}$$

We establish by differentiation that the function $F(b^*)$ increases monotonically and $F(0) = 0, F(\infty) = \infty$. This implies that for any fixed value of the ordinate β of the drain sink B the parameter b^* and the limiting admissible value of the drain flow rate $Q^* = Q(b^*)$ can be determined uniquely from relations (12) and (9). Further, using (8) we can conclude that when β has the same value, the parameters b and r can be found uniquely from relations (5) and (4) for any $Q \in (0, Q^*)$. We note that $b(Q) \in (b_0, b^*)$ and we find the value of $b_0 = \operatorname{sh}^2(\pi\beta/2)$ from (5) when $Q = 0$.

Determination of the mapping parameters enables us to use the analytic relations obtained to compute the flow characteristics and the position of the free boundary CD. Its complex parametric equation can be written using relations (2) and Eq.(4), as

$$z = Q_0 + \frac{2Q}{\pi}\operatorname{arctg}\sqrt{\frac{b}{\zeta}} + i\frac{2}{\pi}\nu,\quad \nu = Q\operatorname{arsh}\sqrt{\frac{b(\zeta-1)}{b+\zeta}} + Q\operatorname{arch}\sqrt{\zeta},\quad \zeta \geqslant 1 \tag{13}$$

When the flow is partially intercepted, relation (13) can be used in computing the dividing stream line R_0R along which $\psi = -Q$. Using the first equation of (2) to write this relation in expanded form, we arrive at the relation

$$(\pi/2)Q_0 - \operatorname{Im}\nu = 0 \tag{14}$$

Choosing now some set of values of the complex variable ζ satisfying Eq.(14), and substituting them into the right-hand side of (13), we find the coordinates of the corresponding pairs on the line R_0R. The position of its end points R_0 and R is found from the following formulas obtained from relations (2) and (4):

$$x_{R_0} = \frac{2Q}{\pi}\operatorname{arctg}\sqrt{\frac{b}{r_0}},\quad y_R = \frac{2}{\pi}\left[Q\ln\frac{\sqrt{b(1+r)}+\sqrt{r(1+b)}}{\sqrt{b}+\sqrt{r}} + Q_0\operatorname{arsh}\sqrt{r}\right],\quad y_{R_0} = x_R = 0 \tag{15}$$

The parameter r_0 is found from the relation $\omega(R_0) = -iQ$. Using the first relation of (2), we can express it with help of the equation used in the first formula of (15)

$$Q \operatorname{arctg} \sqrt{\frac{(1+b)\, r_0}{b\,(1-r_0)}} + Q_0 \arcsin \sqrt{r_0} = \frac{\pi}{2}\, Q \tag{16}$$

In the critical mode with line R_0R coincides with the free boundary CD whose equation is obtained from (13) when $Q = Q^*$, $b = b^*$, $Q_0 = 0$.

Using the analytic relations obtained, we have constructed and programmed an algorithm for computing certain characteristics of the flow in question, with the parameter b found beforehand from (5). The table gives the results of such computations for two versions 1) $\beta = 2{,}0$ and 2) $\beta = 1{,}0$. In each case the flow is first computed in the critical mode when $Q = Q^*$, the latter determined together with the parameter b^* from relations (9) and (12). Next, the computations are carried out for several values of $Q \in (0, Q^*)$ and the data in the table tabulated accordingly. The upper lines correspond to case 1, and the lower lines to case 2. The quantity p_{B_1} can be regarded as the reduced pressure on the drain contour, modelled by an exponential curve passing through the point B_1 with the ordinate $\beta_1 = \beta - 0.05$ in both cases.

Table 1

№	Q/Q^*	Q	Q_s	Q_0	$-p_{B_1}$	x_{R_0}	y_R	$-p_R$	$\lvert w_A \rvert$
1	0,2	0,206	1,010	0,804	0,206	0,203	2,066	0,188	1,016
		0,243	1,060	0,817	0,239	0,221	1,077	0,206	1,101
2	0,4	0,412	1,018	0,606	0,398	0,402	2,133	0,269	1,029
		0,486	1,112	0,626	0,572	0,415	1,154	0,294	1,191
3	0,6	0,619	1,024	0,405	0,576	0,599	2,200	0,291	1,038
		0,729	1,155	0,426	0,840	0,599	1,232	0,319	1,268
4	0,8	0,825	1,028	0,203	0,728	0,798	2,276	0,251	1,045
		0,972	1,190	0,218	1,083	0,790	1,318	0,277	1,331
5	1,0	1,031	1,031	0	0,858	1	2,455	0	1,050
		1,215	1,215	0	1,300	1	1,521	0	1,378

In addition to the values given in the table, Eq.(13) is used to compute the coordinates of some set of points lying on the free boundary CD, and in case when the flow is partially intercepted, also of the points of the dividing stream line RR_0. Relation (14) is used in the latter procedure. The curves computed in this manner are shown in Fig.2, using solid lines for the first case, and dashed lines for the second case. The numbers on the graph and in the table refer to the same values of Q.

Comparing the flow pattern with its characteristics we see that placing the drains deeper ensures the interception of the same proportion of the downward flow in the first as well as in the second case, but with less vacuuming. At the same time, the deep drainage enhances the filtration much less and upsets its uniformity within the subsurface layer of the soil.

If on the other hand we bring the second version to the same value of $\beta = 2{,}0$, as the first, then according to (1) we will have $l = 2{,}0$. All linear quantities including the flow rates, and the quantities associated with them, will increase by a factor of two. The dashed lines in Fig.2 should be corrected accordingly. The differences between two computational versions mentioned above are now seen to be the result of a change in the distance between the drains. Keeping β_1 fixed we have, for $l = 2.0$, $p_{B_1} = 0.5847; 1.1447; 1.6793; 2.1652; 2.6004$ for $Q/Q^* = 0.2; 0{,}4; 0.6; 0.8; 1.0$ respectively. Comparison with the quantities p_{B_1} for the first version shows that doubling the distance separating the drains requires an approximately threefold increase in the vacuum at the drain contour in order to intercept the same proportion of the filtration flow from the surface.

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REFERENCES

1. POLUBARINOVA-KOCHINA P.YA., Theory of Ground Water Motion. Moscow, Nauka, 1977.
2. EMIKH V.N., On some hydrodynamic models of drainage. PMM 43, 6, 1979.
3. EMIKH V.N., Analysis of two-dimensional steady-state filtration into a soil layer with a strongly permeable foundation. PMM 46, 5, 1982.

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